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PROJECTIONS ONTO TRANSLATION-INVARIANT SUBSPACES OF $L_1(\mathbf{R})$

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ABSTRACT. The complemented translation-invariant subspaces of $L_1(\mathbf{R})$ are characterized. This completes an investigation begun by H. P. Rosenthal.

Introduction. In the memoir [**15**] H. P. Rosenthal showed (among other things) that if a translation-invariant subspace I of $L_1(\mathbf{R})$ is complemented, then the hull of I, $hI = \{t: \hat{f}(t) = 0 \text{ for all } f \in I\}$, is of the form

$$\bigcup_{i=1}^{n} (\alpha_i \mathbf{Z} + \beta_i) \setminus F,$$

where F is a finite set and $\{\alpha_i\}$ and $\{\beta_i\}$ are sequences of real numbers. In this paper we show that a necessary and sufficient condition for I to be complemented is that $hI = \bigcup_{i=1}^{n}(\alpha_i \mathbf{Z} + \beta_i)\setminus F$ as above with $\{\alpha_i\}$ pairwise rationally dependent.

One consequence of this result is that each uncomplemented ideal with $hI = \bigcup_{i=1}^{n}(\alpha_i \mathbf{Z} + \beta_i)$ is a $\mathfrak{L}_{1,\lambda}$ subspace of $L_1(\mathbf{R})$ which is not isomorphic to l_1 or L_1. We do not know what the relationship is between these $\mathfrak{L}_{1,\lambda}$ spaces and any of the previous constructions of such $\mathfrak{L}_{1,\lambda}$ spaces [**1, 5, 8**].

This paper is divided into three parts. In the first we prove that the ideals with rationally dependent hull are complemented. In the second we complete the charaterization by showing that the existence of a projection onto an ideal with hull containing rationally independent cosets of $\mathbf{Z}$ would imply that l_1 and l_2 are isomorphic. In the third section we explore the isomorphic properties of the ideals as $\mathfrak{L}_{1,\lambda}$ spaces.

We will use standard notation from Banach space theory as may be found in the book of Lindenstrauss and Tzafriri [**11**] and from harmonic analysis as found in the book of Katznelson [**7**]. The reader will also find the introduction and first section of Rosenthal's memoir [**15**] helpful.

Let us note that each of the sets $E = \bigcup_{i=1}^{n}(\alpha_i \mathbf{Z} + \beta_i)\setminus F$ is a set of spectral synthesis because it is closed and discrete. Thus E uniquely determines an ideal $I(E)$ in $L_1(\mathbf{R})$, namely $I(E) = \{f: \hat{f}|_E = 0\}$. Spectral synthesis will play an important role in our argument involving the structure of $I(E)^{\perp}$. The sets E of the above form coincide with the closed sets in the coset ring of $\mathbf{R}$ endowed with the discrete topology. These are exactly the strong Ditkin sets with no interior [**16**].

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1. The complemented ideals. In this section we will establish some general results about direct sum decompositions of $I(E)^{\perp}$ and show that $I(E)$ is complemented in the rationally dependent case. Many of the results in this section are known but a suitable reference does not seem to be available.

DEFINITION. If α and β are real numbers we will say that α and β are rationally dependent if there exist integers k and l such that $k\alpha + l\beta = 0$. We will denote this by $\alpha \equiv \beta$. If there are not such integers we will say that α and β are rationally independent and denote this by $\alpha \not\equiv \beta$.

LEMMA 1.1. *If $E = \bigcup_{i=1}^{k}(\alpha_i \mathbf{Z} + \beta_i)$ where $\{\alpha_i\}$ and $\{\beta_i\}$ are sequences of real numbers, then $E = \bigcup_{j=1}^{m} E_j$, where for each j*

$$E_j = \bigcup_{k=1}^{n_j} (\theta_j \mathbf{Z} + \gamma_{jk}),$$

$\{\gamma_{jk}\}$ is a sequence of real numbers and the numbers $\theta_1, \theta_2, \ldots, \theta_m$ are pairwise rationally independent.

PROOF. Let $\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_m}\}$ be a maximal set of pairwise rationally independent numbers in $\{\alpha_i\}$. For each $j = 1, 2, \ldots, m$, let $E_j = \bigcup_{\alpha_i \equiv \alpha_{i_j}} (\alpha_i \mathbf{Z} + \beta_i)$. For each fixed integer j there exists a real number θ_j' such that $\theta_j' \equiv \alpha_{i_j}$ and integers k_i such that $k_i \theta_j' = \alpha_i$ for each i such that $\alpha_i \equiv \alpha_{i_j}$. Thus $E_j = \bigcup_{\alpha_i \equiv \alpha_{i_j}} (k_i \theta_j' \mathbf{Z} + \beta_i)$.

Let m_j be the least common multiple of the k_i's for i such that $\alpha_i \equiv \alpha_{i_j}$, and let $\theta_j = m_j \theta_j'$. Evidently

$$k_i \theta_j' \mathbf{Z} + \beta_i = \bigcup_{l=1}^{m_j k_i^{-1}} (\theta_j \mathbf{Z} + l k_i \theta_j' + \beta_i)$$

and so each E_j has the required form. $\square$

REMARK. We may (and shall) assume that for fixed j, the cosets $\theta_j \mathbf{Z} + \gamma_{jk}$, $k = 1, 2, \ldots, n_j$, are disjoint.

LEMMA 1.2. *If α and β are real numbers, $\alpha \neq 0$, then $I(\alpha \mathbf{Z} + \beta)$ is complemented in $L_1(\mathbf{R})$ and*

$$I(\alpha \mathbf{Z} + \beta)^{\perp} = \{ e^{i\beta x} g(x) : g \in L_\infty(\mathbf{R}) \text{ and } g \text{ is periodic of period } 2\pi\alpha^{-1} \}.$$

PROOF. Define $T\colon L_1(\mathbf{R}) \to L_1(\mathbf{R})$ by $Tf(x) = \sum_{n \in \mathbf{Z}} f(x + 2\pi\alpha^{-1} n) 1_{[0, 2\pi\alpha^{-1}]}(x)$. The projection P into $I(\alpha \mathbf{Z} + \beta)$ is then $Pf(x) = f(x) - e^{i\beta x} T(e^{-i\beta s} f(s))(x)$. Indeed, if n is an integer,

$$\begin{aligned}
\widehat{(I - P)} f(\alpha n + \beta) &= \int_{-\infty}^{\infty} e^{i\beta x} T\big(e^{-i\beta s} f(s)\big)(x) e^{-i(\alpha n + \beta)x}\, dx \\
&= \sum_{k \in \mathbf{Z}} \int_0^{2\pi\alpha^{-1}} e^{i\beta(x + 2\pi\alpha^{-1}k)} f(x + 2\pi\alpha^{-1}k) e^{-i\alpha n x}\, dx \\
&= \sum_{k \in \mathbf{Z}} \int_{2\pi\alpha^{-1}k}^{2\pi\alpha^{-1}(k+1)} e^{-i\beta t} f(t) e^{-i\alpha n t} e^{i2\pi k n}\, dt \\
&= \hat{f}(\alpha n + \beta).
\end{aligned}$$

Consequently $\widehat{Pf}(\alpha n + \beta) = 0$ for all $n \in \mathbf{Z}$. Moreover it is easily seen that $I - P$ is a projection of $L_1(\mathbf{R})$ onto $L_1[0, 2\pi\alpha^{-1}]$, regarded as a subspace of $L_1(\mathbf{R})$. Since $\operatorname{supp}(I - P)f \subseteq [0, 2\pi\alpha^{-1}]$, $(I - P)f = 0$ if and only if $f \in I(\alpha n + \beta)$, by the uniqueness of the Fourier series. Hence P is the required projection.

For the second assertion a simple application of the Hahn-Banach theorem proves that

$$I(\alpha\mathbf{Z} + \beta)^{\perp} = \overline{[e^{i(\alpha n + \beta)x} : n \in \mathbf{Z}]}^{w^*} = e^{i\beta x}\overline{[e^{i\alpha n x} : n \in \mathbf{Z}]}^{w^*}.$$

Clearly a w^* limit of $2\pi\alpha^{-1}$-periodic functions is $2\pi\alpha^{-1}$-periodic, proving the result. □

REMARK. It follows from Lemma 2.2 and Corollary 1.3 of **[15]** that if $E = \bigcup_{j=1}^{n}(\alpha\mathbf{Z} + \alpha q_j + \beta)$, where $\alpha, \beta \in \mathbf{R}$ and $\{q_j\} \subseteq Q$, then $I(E)$ is complemented. Indeed, there is a rational number q such that $q^{-1}q_j \in \mathbf{Z}$ for $j = 1, 2, \ldots, n$, and thus $E \subset \alpha q\mathbf{Z} + \beta$. By identifying $[0, 2\pi(\alpha q)^{-1}]$ with $\mathbf{T}$, the circle group, we see that E determines an ideal $\mathcal{I}$ in $L_1(\mathbf{T})$ with $h\mathcal{I}$ in the coset ring of $\mathbf{Z}$. Thus by **[15**, Corollary 1.3] and the remark following, $\mathcal{I}$ is complemented in $L_1(\mathbf{T})$. Finally $L_1(\mathbf{R}) = I(E) \oplus X$, where $X \subseteq L_1[0, 2\alpha(\pi q)^{-1}]$ is the complement of $\mathcal{I}$.

Our next lemma is a consequence of the fact that $E = \bigcup_{i=1}^{k}(\alpha_i\mathbf{Z} + \beta_i)$ is a strong Ditkin set, however we will not use the terminology (see **[9]** for the definition). Applying the results of **[9]** to the set $E = \bigcup_{i=1}^{k}(\alpha_i\mathbf{Z} + \beta_i)\backslash F$, F finite, we get the following

LEMMA 1.3. *Let $E = \bigcup_{i=1}^{k}(\alpha_i\mathbf{Z} + \beta_i)\backslash F$. Then there is a sequence $\{\mu_n\}$ of measures on* $\mathbf{R}$ *such that*:

(i) $\hat{\mu}_n = \mathbf{1}$ *on a neighborhood of* E;
(ii) $\|\mu_n * f\| \to 0$ *as* $n \to \infty$ *for all* $f \in I(E)$;
(iii) $\|\mu_n\| \leq K < \infty$ *for all* n;
(iv) *if* $\rho \notin E$, *then* $\hat{\mu}_n(\rho) = 0$ *for* n *sufficiently large*.

LEMMA 1.4. *Let*

$$E = \bigcup_{j=1}^{m}\bigcup_{k=1}^{n_j}\left((\theta_j\mathbf{Z} + \gamma_{jk})\backslash F\right),$$

where $(\theta_j\mathbf{Z} + \gamma_{jk}) \cap (\theta_l\mathbf{Z} + \gamma_{ls}) \subset F$ if $j \neq l$ or $k \neq s$. Then $I(E)^{\perp}$ is complemented in $L_\infty(\mathbf{R})$ and

$$I(E)^{\perp} = \sum_{j=1}^{m}\sum_{k=1}^{n_j} \oplus\, I\left((\theta_j\mathbf{Z} + \gamma_{jk})\backslash F\right)^{\perp}.$$

PROOF. Fix j and k. Let $\{\mu_n\}$ be the sequence of measures given by Lemma 1.3 for the set $(\theta_j\mathbf{Z} + \gamma_{jk})\backslash F$, and define operators P_n: $L_1(\mathbf{R}) \to L_1(\mathbf{R})$ by $P_n f = \mu_n * f$. By (ii) of Lemma 1.3, $\|P_n f\| \to 0$ as $n \to \infty$ for all $f \in I((\theta_j\mathbf{Z} + \gamma_{jk})\backslash F)$. Let P be a w^* operator limit of P_n^*. Then $\langle Pf, g\rangle = \lim_{n\to\infty}\langle f, P_n g\rangle = 0$, if $g \in I((\theta_j\mathbf{Z} + \gamma_{jk})\backslash F)$. Hence $Pf \in I((\theta_j\mathbf{Z} + \gamma_{jk})\backslash F)^{\perp}$ for all $f \in L_\infty(\mathbf{R})$. If $f \in \operatorname{span}\{e^{i(\theta_j n + \gamma_{jk})x} : n \in \mathbf{Z}\}$, then $\langle Pf, g\rangle = \lim_n\langle P_n^* f, g\rangle = \langle f, g\rangle$ for all $g \in L_1(\mathbf{R})$. Thus range $P = I((\theta_j\mathbf{Z} + \gamma_{jk})\backslash F)^{\perp}$ and $P\,|_{\text{range}\,P} = I\,|_{\text{Range}\,P}$. On the other hand, if

$f \in \operatorname{span}\{e^{i\rho x} : \rho \in E \setminus (\theta_j \mathbf{Z} + \gamma_{jk})\}$, then $\langle Pf, g \rangle = \lim_n \langle f, P_n g \rangle = 0$ for all $g \in L_1(\mathbf{R})$, by (iv) of Lemma 1.3. Now if $f \in I(E \setminus (\theta_j \mathbf{Z} + \gamma_{jk}))^\perp$ and K_l is a summability kernel with $\operatorname{supp} \hat{K}_l \subseteq (-l, l)$, then

$$\begin{aligned}\langle Pf, g \rangle &= \lim_n \langle f, P_n g \rangle = \lim_n \lim_l \langle f, P_n(K_l * g) \rangle \\ &= \lim_l \lim_n \langle f, P_n(K_l * g) \rangle = 0.\end{aligned}$$

It follows that

$$\begin{aligned}\ker P \cap I(E)^\perp &= \left\{ f \in I(E)^\perp : \operatorname{supp} \hat{f} \setminus (\theta_j \mathbf{Z} + \gamma_{jk}) = \varnothing \right\} \\ &= \left\{ f \in I(E)^\perp : \operatorname{supp} \hat{f} \subseteq (\theta_j \mathbf{Z} + \gamma_{jk}) \right\} \\ &= I\left(E \setminus (\theta_j \mathbf{Z} + \gamma_{jk})\right)^\perp,\end{aligned}$$

because these are sets of spectral synthesis. Clearly this implies that

$$I(E)^\perp = \sum_{j=1}^{m} \sum_{k=1}^{n_j} \oplus I\left((\theta_j \mathbf{Z} + \gamma_{jk}) \setminus F\right)^\perp$$

and if we let P_{jk} denote the projection constructed above for $(\theta_j \mathbf{Z} + \gamma_{jk}) \setminus F$, $k = 1, 2, \ldots, n$, then $\sum_{j=1}^{m} \sum_{k=1}^{n_j} P_{jk}$ is a projection onto $I(E)^\perp$.

(The fact that $I(E)^\perp$ is complemented was previously obtained by Gilbert [**3**].) □

REMARK. The nature of the direct sum in the preceding lemma can be determined in some cases. For example, if $E = \bigcup_{j=1}^{m} (\theta_j \mathbf{Z} \setminus \{0\})$ and $\{\theta_j\}$ is independent over the rationals, then the direct sum is in fact equivalent to an l_1 sum (see [**7**, p. 185]). On the other hand, if $E = \bigcup_{j=1}^{m} (\mathbf{Z} + 2\pi j)$, then

$$\sup \left\{ \left\| \sum_{j=1}^{m} \varepsilon_j e^{i2\pi jt} \right\|_\infty : |\varepsilon_j| = 1,\ j = 1, \ldots, m \right\} = m,$$

but $\inf\{\|\sum_{j=1}^{m} \varepsilon_j e^{i2\pi jt}\|_\infty : |\varepsilon_j| = 1, \ldots, m\} = o(m)$. Indeed the Rudin-Shapiro polynomials [**7**, p. 33] yield $O(\sqrt{m})$, and recently Kahane [**6**] has given precise estimates. Y. Gordon and S. Reisner [**4**] have informed us of some related results.

Our next lemma is a simple consequence of standard results from Banach space theory.

LEMMA 1.5. *If $E \subseteq \mathbf{R}$, E closed, and $F \subseteq \mathbf{R}$, F finite, then $I(E)^\perp$ is w^* complemented in $L_\infty(\mathbf{R})$ if and only if $I(E \cup F)^\perp$ is w^* complemented in $L_\infty(\mathbf{R})$.*

Our last lemma gives a method for building commuting projections in the rationally dependent case.

LEMMA 1.6. *Let $E = \bigcup_{i=1}^{k} (\theta \mathbf{Z} + \beta_i)$, where $(\theta \mathbf{Z} + \beta_i) \cap (\theta \mathbf{Z} + \beta_j) = \varnothing$ for $i \neq j$. Then*:

(a) *For each i, there is a measure $\mu_i \in M(\mathbf{R})$ such that*

$$\hat{\mu}_i(y) = \begin{cases} 1 & \text{if } y \in \theta \mathbf{Z} + \beta_i, \\ 0 & \text{if } y \in \theta \mathbf{Z} + \beta_j, j \neq i; \end{cases}$$

(b) *for each i there is a w^* continuous projection P_i^* of $L_\infty(\mathbf{R})$ onto $I(\theta\mathbf{Z} + \beta_i)$ such that $I(\theta\mathbf{Z} + \beta_j)^\perp \subseteq \ker P_i^*$ for $j \neq i$.*

PROOF. (a) Fix i. For each $j \neq i$ let

$$v_j = \frac{1}{1 - e^{2\pi\theta^{-1}i(\beta_i - \beta_j)}}\left[\delta_0 - e^{-2\pi\theta^{-1}i\beta_j}\delta_{-2\pi\theta^{-1}}\right].$$

Then

$$\hat{v}_j(x) = \frac{1 - e^{2\pi\theta^{-1}i(x-\beta_j)}}{1 - e^{2\pi\theta^{-1}i(\beta_i - \beta_j)}} = \begin{cases} 1 & \text{if } x \in \theta\mathbf{Z} + \beta_i, \\ 0 & \text{if } x \in \theta\mathbf{Z} + \beta_j, j \neq i. \end{cases}$$

Let $\mu_i = v_1 * v_2 * \cdots * v_{i-1} * v_{i+1} * \cdots * v_k$. Clearly μ_i satisfies (a).

For (b), let Q_i^* be the w^* continuous projection of $L_\infty(\mathbf{R})$ onto $I(\theta\mathbf{Z} + \beta_i)^\perp$ given by Lemma 1.2 and define $P_i^* f = Q_i^*(\mu_j * f)$ for $f \in L_\infty(\mathbf{R})$. Just as in the proof of Lemma 1.4,

$$\mu_i * f = \begin{cases} f & \text{if } f \in I(\theta\mathbf{Z} + \beta_i)^\perp, \\ 0 & \text{if } f \in I(\theta\mathbf{Z} + \beta_j)^\perp, \text{ if } j \neq i. \end{cases}$$

Hence

$$P_i^*(f) = \begin{cases} f & \text{if } f \in I(\theta\mathbf{Z} + \beta_i)^\perp, \\ 0 & \text{if } f \in I(\theta\mathbf{Z} + \beta_j)^\perp, \text{ if } j \neq i, \end{cases}$$

as required. □

We remark that part (b) is essentially contained in Theorem 5 of **[12]**.

We are now ready to prove the main result of this section.

THEOREM A. *Suppose $E = \bigcup_{i=1}^k (\theta_i\mathbf{Z} + \gamma_i)\setminus F$, where F is a finite set, $\{\theta_i\}$ and $\{\gamma_i\}$ are sequences of real numbers and $\theta_i \equiv \theta_j$ for all i and j. Then $I(E)$ is a complemented subspace of $L_1(\mathbf{R})$.*

PROOF. First note that $I(E)$ is complemented in $L_1(\mathbf{R})$ if and only if $I(E)^\perp$ is w^* complemented in $L_\infty(\mathbf{R})$. By Lemma 1.5 we may assume that $F = \varnothing$ and by Lemma 1.1 that $E = \bigcup_{i=1}^n (\theta\mathbf{Z} + \beta_i)$, where $(\theta\mathbf{Z} + \beta_i) \cap (\theta\mathbf{Z} + \beta_j) = \varnothing$ if $i \neq j$. Finally by Lemma 1.6(b) there is a sequence of w^* continuous projections $\{P_i\}$ such that for each i, P_i projects $L_\infty(\mathbf{R})$ onto $I(\theta\mathbf{Z} + \beta_i)^\perp$ and $\ker P_i \supseteq I(\theta\mathbf{Z} + \beta_j)^\perp$ for $j \neq i$. Hence $P = \sum_{i=1}^n P_i$ is a w^* continuous projection of $L_\infty(\mathbf{R})$ onto $\sum_{i=1}^n I(\theta\mathbf{Z} + \beta_i)^\perp = I(\bigcup_{i=1}^n (\theta\mathbf{Z} + \beta_i))^\perp$. □

2. Uncomplemented ideals. In **[15]** Rosenthal shows that if $E \subseteq \mathbf{R}$ is closed, then a necessary condition for an ideal I with $hI = E$ to be complemented in $L_1(\mathbf{R})$ is that E belong to the coset ring of $\mathbf{R}$ with the discrete topology. He then shows that such an E has the form $\bigcup_{i=1}^k (\alpha_i\mathbf{Z} + \beta_i)\setminus F$, where $\{\alpha_i\}$ and $\{\beta_i\}$ are sequences of real numbers, and F is a finite set. In the previous section we proved that $I(E)$ is complemented if the α_i's are pairwise rationally dependent. In this section we will show that this is a necessary condition.

The proof will be by contradiction, that is, we will assume that there is a projection Q of $L_1(\mathbf{R})$ onto $I(E)$, where $E = \bigcup_{i=1}^k(\alpha_i\mathbf{Z} + \beta_i)$ and (say) $\alpha_1 \not\approx \alpha_2$. (Lemma 1.5 allows us to ignore the finite set F.) We will then show that the fact that the cosets $\alpha_1\mathbf{Z} + \beta_1$ and $\alpha_2\mathbf{Z} + \beta_2$ have points arbitrarily close together induces an isomorphism from l_1 to l_2.

Lemma 2.1. *Let $A_1, A_2, \ldots, A_n$ be subspaces of $L_1(\mathbf{R})$ such that $X = [A_i^\perp : i = 1, 2, \ldots, n]$ is w^* complemented in $L_\infty(\mathbf{R})$ and $X = \sum_{i=1}^n \oplus A_i^\perp$. Then there is a projection P of $L_1(\mathbf{R})$ onto A_1 such that* $\ker P \subseteq \bigcap_{i=2}^n A_i$.

Proof. Each of the subspaces $A_i^\perp$ of $L_\infty(\mathbf{R})$ is w^* closed and thus X is a w^* direct sum of the $A_i^\perp$'s. Therefore there is a w^* continuous projection R^* of X onto $A_1^\perp$ such that $\ker R^* \supseteq \bigcup_{i=2}^n A_i^\perp$. Let Q^* be the w^* continuous projection of $L_\infty(\mathbf{R})$ onto X and let $P^* = I - R^*Q^*$.

If $f \in A_1$ and $g \in L_\infty(\mathbf{R})$, then

$$\langle g, Pf\rangle = \langle (I - R^*Q^*)g, f\rangle = \langle g, f\rangle - \langle R^*Q^*g, f\rangle = \langle g, f\rangle.$$

Hence $P|_{A_1} = I|_{A_1}$ and clearly $P(L_1(\mathbf{R})) \subseteq A_1$.

If $h \in \ker P$ and $g \in L_\infty(\mathbf{R})$, then $0 = \langle g, Ph\rangle = \langle (I - R^*Q^*)g, h\rangle$. If now $g \in A_i^\perp$, for some $i = 2, 3, \ldots, n$, then $R^*Q^*g = R^*g = 0$. Thus $\langle g, h\rangle = 0$, i.e., $h \in \bigcap_{i=2}^n (A_i^\perp)_\perp = \bigcap_{i=2}^n A_i$. □

For the remainder of this section, $S = S(\alpha)$ will denote the operator from $L_1(\mathbf{R})$ onto $L_1[0, 2\pi\alpha^{-1}]$ defined by

$$Sf(x) = \sum_{n\in\mathbf{Z}} f(x + 2\pi\alpha^{-1}n)1_{[0,2\pi\alpha^{-1}]}(x).$$

Note that S is a projection onto $L_1[0, 2\pi\alpha^{-1}]$, $\widehat{Sf}(\alpha n) = \hat{f}(\alpha n)$ for all $n \in \mathbf{Z}$ and $f \in L_1(\mathbf{R})$, and $\ker S = I(\alpha\mathbf{Z})$. (See the proof of Lemma 1.2.)

Lemma 2.2. *If P is a projection from $L_1(\mathbf{R})$ onto $I(\alpha\mathbf{Z})$, then $S|_{\ker P}$ is an isomorphism onto $L_1[0, 2\pi\alpha^{-1}]$.*

Proof. The projection S defines a decomposition $L_1(\mathbf{R}) = I(\mathbf{Z}) \oplus L_1[0, 2\pi]$ and the projection P defines another decomposition $L_1(\mathbf{R}) = I(\mathbf{Z}) \oplus X$ where $X = \ker P$. Because $\ker S = I(\mathbf{Z})$, $S|_X$ is one-to-one. The range of $S|_X$ equals $S(L_1(\mathbf{R})) = L_1[0, 2\pi]$. Hence $S|_X$ is an isomorphism onto $L_1[0, 2\pi]$ as claimed. □

The next lemma is really the heart of the proof that $I(E)$ is not complemented when E contains rationally independent cosets.

Lemma 2.3. *There is no projection P of $L_1(\mathbf{R})$ onto $I(\alpha\mathbf{Z})$ such that* $\ker P \subseteq$ *$I((\alpha'\mathbf{Z} + \beta)\setminus F)$ where $\alpha \not\approx \alpha'$ and F is a finite set.*

Proof. Suppose P is such a projection. Then by Lemma 2.2, if $X = \ker P$, $S|_X$ is an isomorphism onto $L_1[0, 2\pi\alpha^{-1}]$. Let $x_n = (S|_X)^{-1}e^{i\alpha nx}1_{[0,2\pi\alpha^{-1}]}(x)$, $n = 1, 2, \ldots$.

Claim. For every $k \in \mathbf{N}$ there is an $n \in \mathbf{N}$ such that $\|x_n - x_n1_{[-k,k]}\| \geq \pi\alpha^{-1}/2$.

First observe that if $t_1, t_2 \in \mathbf{R}$, then

$$\|\widehat{x_n1_{[-k,k]}}(t_1) - \widehat{x_n1_{[-k,k]}}(t_2)| \leq |k||t_1 - t_2|\|(S|_X)^{-1}\|2\pi\alpha^{-1},$$

because

$$\left|\frac{d}{dt}\widehat{x_n 1_{[-k,k]}}(t)\right| \leqslant |k| \, \|x_n\|.$$

Because $\alpha \not\approx \alpha'$, for every $\delta > 0$ there are integers n_1 and n_2 such that $|\alpha n_1 - (\alpha' n_2 + \beta)| < \delta$ and $\alpha' n_2 + \beta \notin F$. We have that $\hat{x}_{n_1}(an_1) = \widehat{Sx}_{n_1}(\alpha n_1) = 2\pi\alpha^{-1}$ and $\hat{x}_{n_1}(\alpha' n_2 + \beta) = 0$. Thus if δ is sufficiently small, either

$$|\hat{x}_{n_1}(\alpha n_1) - \widehat{x_{n_1} 1_{[-k,k]}}(\alpha n_1)| \geqslant \tfrac{1}{4}(2\pi\alpha^{-1}),$$

or

$$|\hat{x}_{n_1}(\alpha' n_2 + \beta) - x_{n_1} 1_{[-k,k]}(\alpha' n_2 + \beta)| \geqslant \tfrac{1}{4}(2\pi\alpha^{-1}).$$

In either case $\|x_{n_1} - x_{n_1} 1_{[-k,k]}\| > \pi\alpha^{-1}/2 = \rho$, proving the claim.

It follows by a standard gliding hump argument that there is a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ and a strictly increasing sequence of integers $\{k_l\}$ such that

(i) $\|x_{n_l} 1_{[-k_l, -k_{l-1}] \cup [k_{l-1}, k_l]}\| > \rho/2$, and

(ii) $n_l > 3n_{l-1}$

for each $l \in \mathbf{N}$. By Rosenthal's disjointness lemma **[13]**, there is a further subsequence of $\{x_{n_l}\}$, $\{x_{n_l}\}_{l \in M}$, M is infinite, which is equivalent to the usual unit vector basis of l_1. On the other hand $\{Sx_{n_l}\}_{l \in M}$ is a lacunary sequence of exponentials $\{e^{i\alpha n_l x} 1_{[0, 2\pi\alpha^{-1}]}\}_{l \in M}$ and thus is equivalent to the usual unit vector basis of l_2. This contradicts the assertion that S is an isomorphism on X. □

We are now ready to prove

THEOREM B. *Let $E = \bigcup_{i=1}^{k}(\alpha_i \mathbf{Z} + \beta) \setminus F$ where $\{\alpha_i\}$ and $\{\beta_i\}$ are sequences of real numbers, F is a finite set, and the α_i are not pairwise rationally dependent. Then $I(E)$ is not complemented in $L_1(\mathbf{R})$.*

PROOF. By Lemmas 1.1 and 1.5 we may assume that $(\alpha_i \mathbf{Z} + \beta_i) \cap (\alpha_j \mathbf{Z} + \beta_j) \subseteq F$ where $i \neq j$. Hence by Lemma 1.4

$$I(E)^{\perp} = \sum_{i=1}^{k} \oplus I((\alpha_i \mathbf{Z} + \beta_i) \setminus F)^{\perp}.$$

Clearly we may also assume that $\alpha_1 \not\approx \alpha_2$ and that $\beta_1 = 0$.

If $I(E)$ is complemented in $L_1(\mathbf{R})$, it follows that $I(E)^{\perp}$ is w^* complemented in $L_\infty(\mathbf{R})$, and so, by Lemma 2.1, there is a projection P of $L_1(\mathbf{R})$ onto $I(\alpha_1 \mathbf{Z} \setminus F)$ with $\ker P \subseteq I((\alpha_2 \mathbf{Z} + \beta_2) \setminus F)$. Because F is finite, $I(\alpha_1 \mathbf{Z} \setminus F)/I(\alpha_1 \mathbf{Z})$ is finite dimensional. (Since $I(\alpha_1 \mathbf{Z} \setminus F) \cap I((\alpha_2 \mathbf{Z} + \beta_2) \setminus F)$ contains a copy of L_1, and $I(\alpha_1 \mathbf{Z} \setminus F) \cap I((\alpha_2 \mathbf{Z} + \beta_2) \setminus F) \cap \ker P = \{0\}$, it follows that $I(\alpha_2 \mathbf{Z} \setminus F)/\ker P$ is finite dimensional. See Proposition 3.1.) Hence we may assume that P is a projection onto $I(\alpha_1 \mathbf{Z})$ such that $\ker P \subseteq I((\alpha_2 \mathbf{Z} + \beta_2) \setminus F)$. But this contradicts Lemma 2.3 with $\alpha = \alpha_1$, $\alpha' = \alpha_2$, and the theorem follows. □

3. The ideals as $\mathcal{L}_{1,\lambda}$ spaces. In this section we will show that if

$$E = \bigcup_{i=1}^{k} (\alpha_i \mathbf{Z} + \beta_i) \setminus F,$$

then $I(E)$ is a $\mathcal{L}_{1,\lambda}$ space and contains a complemented isomorphic copy of L_1. These are not new results, but combining them with the results of the previous section gives a possibly new set of $\mathcal{L}_{1,\lambda}$ spaces.

PROPOSITION 3.1. *Let $E = \bigcup_{i=1}^{k}(\alpha_i \mathbf{Z} + \beta_i)\setminus F$ where $\{\alpha_i\}$ and $\{\beta_i\}$ are sequences of real numbers, F is a finite set, and $\{\alpha_i\}$ are not pairwise rationally dependent. Then*

(i) *L_1 is isomorphic to a complemented subspace of $I(E)$,*

(ii) *$I(E)$ is a $L_{1,\lambda}$ space,*

(iii) *$I(E)$ is not isomorphic to L_1.*

PROOF. We may assume that $\alpha_i > 0$, $i = 1,2,\ldots,k$. For each $i \leqslant k$, let $\mu_i = \delta_0 - e^{-\alpha_i^{-1}2\pi\beta_i}\delta_{-2\pi\alpha^{-1}}$. Define an operator T on $L_1(\mathbf{R})$ by $Tf = \mu_1 * \mu_2 * \cdots * \mu_k * f$. If $0 < s < \min 2\pi\alpha_i^{-1}$, then $T\big|_{L_1[0,s]}$ is an isomorphism and $TL_1(\mathbf{R}) \subset I(E)$. Indeed $\mu_i * g(x) = g(x) - e^{-i\alpha_i^{-1}2\pi\beta_i}g(x + 2\pi\alpha_i^{-1})$, and thus if $\operatorname{supp} f \subseteq [0, s]$, $(\mu_i * f)\big|_{[0,s]} = f$, while if $\operatorname{supp} f \subseteq (s, \infty)$, $(\mu_i * f)\big|_{[0,s]} = 0$. Hence $\|Tf\| \geqslant \|f\|$ for $f \in L_1[0, s]$. To see that $TL_1(\mathbf{R}) \subseteq I(E)$ it suffices to note that

$$\widehat{Tf}(x) = \prod_{i=1}^{k}\left(1 - e^{i\alpha_i^{-1}2\pi(x-\beta_i)}\right)\hat{f}(x).$$

Moreover $TL_1[0, s]$ is complemented by the projection $Pf = T(f\big|_{[0,s]})$.

To see (ii), it follows from Lemma 1.4 that $I(E)^{\perp}$ is complemented in $L_\infty(\mathbf{R})$ and so $I(E)^* = L_\infty(R)/I(E)^{\perp}$ is isomorphic to the complement of $I(E)^{\perp}$. It is well known that every complemented subspace of $L_\infty(\mathbf{R})$ is isomorphic to $L_\infty(\mathbf{R})$ [**11**, p. 57]. Thus $I(E)^*$ is a $\mathcal{L}_{\infty,\lambda}$ space and hence $I(E)$ is a $\mathcal{L}_{1,\lambda}$ space [**10**].

Finally by the results of §2, $I(E)$ is not complemented. Because $L_1(\mathbf{R})/I(E)$ is a $\mathcal{L}_{1,\lambda}$ space, a result of Lindenstrauss [**9**] implies that $I(E)$ is not complemented in a second conjugate space. Thus $I(E)$ is not isomorphic to L_1.

All of the previous constructions [**1, 5, 8**] of $\mathcal{L}_{1,\lambda}$ spaces gave examples which do not contain L_1 and so are not isomorphic to the ideals above. However it is possible that the ideals are simply direct sums of L_1 with the previously known examples.

We ask the following questions:

1. If $\alpha_1 \not\approx \alpha_2$, is $I(\alpha_1\mathbf{Z} \cup \alpha_2\mathbf{Z})$ isomorphic to $I(\mathbf{Z} \cup \sqrt{2}\mathbf{Z})$?
2. More generally, what is the isomorphic classification of the ideals

$$I\left(\bigcup_{i=1}^{k}(\alpha_i\mathbf{Z} + \beta_i)\right)?$$

The authors with J. M. Rosenblatt have obtained some extensions of these results to $L_1(G)$ where G is a locally compact abelian group. These results will appear in a forthcoming paper.

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